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Nambu–Jacobi structures and Jacobi algebroids

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Abstract

We describe a Nambu–Jacobi structure as a 'Nambu–Poisson' structure on a certain Jacobi algebroid. It is shown that the matched pair of Leibniz algebroids for a Nambu–Jacobi structure is the Leibniz algebroid associated with this 'Nambu–Poisson' structure. We also see that a different Leibniz algebroid is associated with a Nambu–Jacobi structure, which is a natural generalization of the Lie algebroid associated with a Jacobi structure.

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1. Introduction

The aim of this paper is to understand a Nambu–Jacobi structure as a deformed 'Nambu–Poisson' structure by means of differential calculus on Lie algebroids in the presence of a one-cocycle. It is motivated from the results in [17] and [10] for a Jacobi structure.

The Jacobi structure [21], which gives a unified approach for two important notions of the Poisson structure and the contact structure, has been studied in many papers such as [2, 12, 19, 20, 25, 31]. In [17] it was shown that, using analogues of the Schouten bracket, the Lie derivative and the exterior differential, one can describe formally a Jacobi structure and its associated Lie algebroid structure on the bundle of 1-jets as a Poisson structure and its associated Lie algebroid structure on the cotangent bundle. In [10] the 'deformed' bracket and operations above were treated as the natural bracket and operations on *the Jacobi algebroid*. A Jacobi algebroid is equivalent to a Lie algebroid endowed with a one-cocycle. It is also equivalent to a *Schouten–Jacobi bracket* for first-order differential operators, which will be suitable for the description of frame-free mechanics. In [32], the Jacobi structure was generalized in the direction of *the Dirac structure* [3, 4], which was called *the* $\mathcal{E}^1(M)$*-Dirac structure* and the associated foliation was investigated in [18]. As the Lie algebroid and the Dirac structure are becoming recognized as the basic structures for the description of dynamical systems with or without constraints, it would be of interest to investigate the 'Dirac structure' on a Jacobi algebroid further.

A Nambu–Jacobi structure [14, 16, 26, 27] is an extension of the notion of a Jacobi structure. It is defined by a multilinear skew-symmetric bracket on the space of smooth functions which is a first-order differential operator as in the case of a Jacobi structure, and satisfies *the fundamental identity*

$$\{f_1, \dots, f_{p-1}, \{g_1, \dots, g_p\}\} = \sum_{i=1}^{p} \{g_1, \dots, \{f_1, \dots, f_{p-1}, g_i\}, \dots, g_p\}$$

for all $f_1, \dots, f_{p-1}, g_1, \dots, g_p \in C^\infty(M)$. When the bracket satisfies the Leibniz rule

$$\{g_1 g_2, f_1, \dots, f_{p-1}\} = g_1\{g_2, f_1, \dots, f_{p-1}\} + g_2\{g_1, f_1, \dots, f_{p-1}\},$$

it is nothing but a Nambu–Poisson structure [29]. The equivalent definition of order $p \geqslant 3$ using multivector fields is given in [11, 15, 27].

In recent years, Nambu–Poisson structures have received much attention in both physics and mathematics [5–9, 11, 13, 15, 26, 28, 30, 33]. A description of a Nambu–Jacobi structure in terms of the Jacobi algebroid will be of help in generalizing the theories using derivations to those using first-order differential operators.

Although it is a natural extension, there are some differences between the Nambu–Jacobi structure of order 2 (that is, the Jacobi structure) and that of order greater than 2. For example, a Nambu–Jacobi structure of order $p \geqslant 3$ has the pair of associated foliations whose leaves are of dimension p, $p-1$ or zero and is endowed with volume forms, while the foliation associated with a Jacobi structure consists of contact or locally conformal symplectic leaves. Besides, it was proved in [16] that a Nambu–Jacobi structure of order greater than 2 associates a *Leibniz algebroid*, which is a non-commutative version of the Lie algebroid and it is not generalized to the case of order two.

In this paper, we describe a Nambu–Jacobi structure on a manifold as a 'Nambu–Poisson structure' using the notion of a Jacobi algebroid as in the case of order 2. We also see that a Nambu–Jacobi structure gives another Leibniz algebroid which is a natural generalization of the Lie algebroid associated with a Jacobi structure (theorem 3.6). The Jacobi algebroid is obtained by choosing the one-cocycle $(0, p-1)$ consisting of zero 1-form and a constant function $p-1$. It gives simple descriptions of both the definition of a Nambu–Jacobi structure using multivector fields and the associated Leibniz algebroid structure; in fact, they formally correspond to those for the Nambu–Poisson structure [15]. The new Leibniz algebroid structure which we obtain is a formal counterpart of that associated with a Nambu–Poisson structure given in [11]. This will lead us to the generalization of *the Nambu-Dirac manifold* [11] in the direction of the Nambu–Jacobi manifold. The Nambu-Dirac structure is an extension of the Dirac structure to higher orders, and it contains the Dirac structure, the Nambu–Poisson structure and the multisymplectic structure in the sense of [1] (a similar extension is proposed in [33], which is the generalized strong Nambu-Dirac structure in [11]). The generalization will be an extension of $\mathcal{E}^1(M)$-Dirac structure to higher orders.

This paper is organized as follows. In section 2, we review the basic definitions and properties of Nambu–Jacobi structures and Leibniz algebroids. We also recall the differential calculus on Lie algebroids in the presence of a one-cocycle and its applications to Jacobi structures. In section 3, we interpret a Nambu–Jacobi structure as a 'Nambu–Poisson structure' on a certain Jacobi algebroid and give a simple description of the associated Leibniz algebroid. We also see that a different Leibniz algebroid is associated with a Nambu–Jacobi structure, which is a natural generalization of the Lie algebroid associated with a Jacobi structure. Finally we discuss a generalization of our results from the case of manifolds to that of Lie algebroids.

It still remains unclear why the one-cocycle $(0, p-1)$ plays a special role for Nambu–Jacobi structures of order p. It may be the clue that

$$(\Pi, E)(\mathrm{d}^{(0,1)} f_1 \wedge \cdots \wedge \mathrm{d}^{(0,1)} f_{p-1}) = (\Pi, E)(\mathrm{d}^{(0,p-1)} \boldsymbol{f})$$

where

$$\boldsymbol{f} = \left(\frac{1}{p-1} \sum_{i=1}^{p} (-1)^{i-1} f_i \, \mathrm{d}f_1 \wedge \cdots \wedge \widehat{\mathrm{d}f_i} \wedge \cdots \wedge \mathrm{d}f_{p-1}, 0 \right).$$

We will return to this question in the future.

2. Definitions and known facts

2.1. Nambu–Jacobi structures and Leibniz algebroids

First, we review the notions of Nambu–Jacobi structure and Leibniz algebroid.

A *Nambu–Jacobi structure of order p* on a manifold $M (2 \leqslant p \leqslant \dim M)$ is a p-linear skew-symmetric map $\{,\ldots,\} : C^\infty(M) \times \cdots \times C^\infty(M) \to C^\infty(M)$ satisfying

(i) (first-order differential operation)

$$\{g_1 g_2, f_1, \ldots, f_{p-1}\} = g_1\{g_2, f_1, \ldots, f_{p-1}\} + g_2\{g_1, f_1, \ldots, f_{p-1}\} - g_1 g_2\{1, f_1, \ldots, f_{p-1}\}$$

(ii) (fundamental identity)

$$\{f_1, \ldots, f_{p-1}, \{g_1, \ldots, g_p\}\} = \sum_{i=1}^{p} \{g_1, \ldots, \{f_1, \ldots, f_{p-1}, g_i\}, \ldots, g_p\}$$

for all $f_1, \ldots, f_{p-1}, g_1, \ldots, g_p \in C^\infty(M)$. We call a manifold M endowed with such a bracket a *Nambu–Jacobi manifold of order p*. A Nambu–Jacobi manifold of order 2 is nothing but *a Jacobi manifold* and *a Nambu–Poisson manifold* is a Nambu–Jacobi manifold whose bracket vanishes if one of the functions is constant.

A Nambu–Jacobi manifold $(M, \{,\ldots,\})$ of order p is equivalently defined as the pair of a p-vector field Π and a $(p-1)$-vector field E. We have

$$[\Pi, \Pi] = -2E \wedge \Pi \qquad [E, \Pi] = 0$$

for $p = 2$, that is, a Jacobi structure. For $p \geqslant 3$ we have (the condition of the following theorem is slightly restricted in [11], which is easily loosened)

Theorem 2.1 [11]. *Let Π be a p-vector field and E a $(p-1)$-vector field where $p \geqslant 2$, from which we assume Π is locally decomposable (that is, $\Pi = X_1 \wedge \cdots \wedge X_p$ for some vector fields $X_1, \ldots, X_p$ around a point where $\Pi \neq 0$) when $p = 2$. The pair (Π, E) defines a Nambu–Jacobi structure if and only if*

$$[\Pi(\alpha), \Pi] = (-1)^p (\Pi(\mathrm{d}\alpha))\Pi \tag{1}$$

$$[E(\beta), E] = (-1)^{p-1} (E(\mathrm{d}\beta))E \tag{2}$$

$$[E(\beta), \Pi] = (-1)^{p-1} (E(\mathrm{d}\beta))\Pi \tag{3}$$

$$[\Pi(\alpha), E] = (-1)^p (\Pi(\mathrm{d}\alpha))E + (-1)^{p-1} \Pi(\mathrm{d}(E(\alpha))) \tag{4}$$

for any $(p-1)$-form α and $(p-2)$-form β.

We remark that we recover the definition of a Nambu–Poisson structure when $E = 0$. In fact, if the order p of a Nambu–Jacobi structure (Π, E) is greater than 2, then Π and E are Nambu–Poisson structures of orders p and $p-1$, respectively (see [14, 27]).

Example 2.2.

(i) Let M be an n-dimensional manifold, ν its co-volume field (that is, an n-vector field) and f an arbitrary function. The pair $(\nu, \nu(\mathrm{d}f))$ is a Nambu–Jacobi structure of order n on M.
(ii) Let (M, Π) be a nonsingular Nambu–Poisson manifold of order $p < \dim M$ and υ a vector field transverse to the associated foliation. The pair $(\upsilon \wedge \Pi, \Pi)$ is a Nambu–Jacobi structure of order $p + 1$.

As well as a Nambu–Poisson manifold, a Nambu–Jacobi manifold of order $p \geqslant 3$ associates the structure called the *Leibniz algebroid*. A *Leibniz algebra* $(V, [\![,]\!])$ is an R-module V, where R is a commutative ring, endowed with a bilinear map $[\![,]\!] : V \times V \to V$ satisfying

$$[\![a_1, [\![a_2, a_3]\!]]\!] = [\![[\![a_1, a_2]\!], a_3]\!] + [\![a_2, [\![a_1, a_3]\!]]\!] \tag{5}$$

for $a_1, a_2, a_3 \in V$ (see [22, 23]). The map $[\![,]\!]$ is called the *Leibniz bracket on V* and (5) the *Leibniz identity* (remark that we use the term Leibniz rule for the derivation law). If $[\![,]\!]$ is additionally skew-symmetric, then the Leibniz identity is just the Jacobi identity and $(V, [\![,]\!])$ is a Lie algebra. Therefore, a Leibniz algebra is a non-commutative variant of a Lie algebra. A Leibniz algebroid is defined in the same way by generalizing the notion of a Lie algebroid [24].

Definition 2.3 [15]**.** *A Leibniz algebroid is a smooth vector bundle $\Pi : A \to M$ with a Leibniz algebra structure $[\![,]\!]$ on $\Gamma(A)$ and a bundle map $\rho : A \to TM$, called an anchor, such that the induced map $\rho : \Gamma(A) \to \Gamma(TM)$ satisfies the derivation law*

$$[\![X, fY]\!] = ((\rho(X))f)Y + f[\![X, Y]\!]$$

for all $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$.

We remark that the condition $\rho([\![X, Y]\!]) = [\rho(X), \rho(Y)]$ is included in the original definition [15], which in fact follows from the other conditions of the definition as in the case of a Lie algebroid.

Theorem 2.4 [16]**.** *Let (M, Π, E) be a Nambu–Jacobi manifold of order $p \geqslant 3$. Then the triple $(\wedge^{p-1}T^*M \oplus \wedge^{p-2}T^*M, [\![,]\!], \rho_{(\Pi,E)})$ is a Leibniz algebroid over M where*

$$\begin{aligned}[\![(\alpha, \alpha'), (\beta, \beta')]\!] = (&\mathcal{L}_{\Pi(\alpha)}\beta + (-1)^p(\Pi(\mathrm{d}\alpha))\beta + \mathcal{L}_{E(\alpha')}\beta \\ &+ (-1)^{p-1}(E(\mathrm{d}\alpha'))\beta + (-1)^{p-1}\,\mathrm{d}(E(\alpha)) \wedge \beta', \\ &\mathcal{L}_{E(\alpha')}\beta' + (-1)^{p-1}(E(\mathrm{d}\alpha'))\beta' + \mathcal{L}_{\Pi(\alpha)}\beta' + (-1)^p(\Pi(\mathrm{d}\alpha))\beta')\end{aligned}$$

$$\rho_{(\Pi,E)}(\alpha, \alpha') = \Pi(\alpha) + E(\alpha')$$

for $(\alpha, \alpha'), (\beta, \beta') \in \wedge^{p-1}T^*M \oplus \wedge^{p-2}T^*M$, respectively.

Although this is proved in [16], we will give an alternative proof later. When $E = 0$, we recover the Leibniz algebroid structure on $\wedge^{p-1}T^*M$ associated with the Nambu–Poisson manifold (M, Π) [15].

Two Leibniz algebroids A_1, A_2 over M are said to form a *matched pair of Leibniz algebroids* [16] if the Whitney sum $A = A_1 \oplus A_2$ has a Leibniz algebroid structure such that A_1 and A_2 are Leibniz subalgebroids of A. When (Π, E) is as above, $\wedge^{p-1}T^*M$ and $\wedge^{p-2}T^*M$ have Leibniz algebroid structures associated with the Nambu–Poisson structures Π and E respectively. In fact, the above theorem implies that these two Leibniz algebroids form a matched pair of Leibniz algebroids.

2.2. *Differential calculus on Lie algebroids in the presence of a one-cocycle*

We mainly summarize differential calculus on Lie algebroids in the presence of a one-cocycle and its applications to Jacobi structures which is investigated in [10, 17].

Let $A \to M$ be a vector bundle. For $r \geqslant 1$, we can identify $\Gamma(\wedge^r(A \times \mathbb{R}))$ with $\Gamma(\wedge^r A) \oplus \Gamma(\wedge^{r-1}A)$ by

$$(P, Q)((\alpha_1, f_1), \ldots, (\alpha_r, f_r)) = P(\alpha_1, \ldots, \alpha_r) + \sum_{i=1}^{r}(-1)^{i+1} f_i Q(\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_r)$$

for any $(P, Q) \in \Gamma(\wedge^r A) \oplus \Gamma(\wedge^{r-1}A)$ and $(\alpha_i, f_i) \in \Gamma(A^*) \oplus C^\infty(M)$.

Suppose that $(A, [,], \rho)$ be a Lie algebroid over a manifold M. The next proposition is a version of the case $TM \times \mathbb{R}$ given in [24].

Proposition 2.5. *The vector bundle $A \times \mathbb{R} \to M$ has a Lie algebroid structure whose bracket and anchor are given by*

$$[(X, f), (Y, g)] = ([X, Y], (\rho(X))g - (\rho(Y))f) \qquad \tilde{\rho}(X, f) = \rho(X).$$

Let $\phi \in \Gamma(A^*)$ be a one-cocycle in the Lie algebroid cohomology complex with trivial coefficients, that is, ϕ satisfies the equation

$$\phi([X, Y]) = (\rho(X))(\phi(Y)) - (\rho(Y))(\phi(X))$$

for all $X, Y \in \Gamma(A)$. It induces the 'ϕ-deformed' action $\rho^\phi : \Gamma(A) \times C^\infty(M) \to C^\infty(M)$ defined by

$$(\rho^\phi(X))f = (\rho(X))f + \phi(X)f$$

for $X \in \Gamma(A)$ and $f \in C^\infty(M)$. Thus we can consider the 'ϕ-deformed' Lie algebroid cohomology with trivial coefficients. The restriction of its coboundary operator to $\Gamma(\wedge^* A^*)$ is called the ϕ-*differential* and denoted by d^ϕ. It satisfies

$$d^\phi\alpha = \mathrm{d}\alpha + \phi \wedge \alpha.$$

The ϕ-*Lie derivative* $\mathcal{L}^\phi_X : \Gamma(\wedge^* A^*) \to \Gamma(\wedge^* A^*)$ with respect to $X \in \Gamma(A)$ is also defined in the usual manner. We have

$$\mathcal{L}^\phi_X f = \iota_X(d^\phi f) \qquad \mathcal{L}^\phi_X\alpha = d^\phi(\iota_X\alpha) + \iota_X(d^\phi\alpha)$$

for $f \in C^\infty(M)$ and $\alpha \in \Gamma(\wedge^k A^*)$. It is an analogue of the usual Cartan formula. However, some of the calculus is deformed by ϕ.

Proposition 2.6.

$$\mathcal{L}^\phi_{fX}\alpha = f\mathcal{L}^\phi_X\alpha + \mathrm{d}f \wedge \iota_X\alpha$$
$$d^\phi(\alpha \wedge \alpha') = d^\phi\alpha \wedge \alpha' + (-1)^k\alpha \wedge d^\phi\alpha' - \phi \wedge \alpha \wedge \alpha'$$
$$\mathcal{L}^\phi_X(\alpha \wedge \alpha') = \mathcal{L}^\phi_X\alpha \wedge \alpha' + \alpha \wedge \mathcal{L}^\phi_X\alpha' - \phi(X)\alpha \wedge \alpha'$$

for $\alpha \in \Gamma(\wedge^k A^*)$, $\alpha' \in \Gamma(\wedge^{k'} A^*)$ and $X \in \Gamma(A)$.

It was also elucidated in [10, 17] that, for a Lie algebroid and its one-cocycle, there also exists the 'ϕ-deformed' Schouten bracket (we follow the convention of signs used in [10]).

Theorem 2.7. *Let* $(A, [,], \rho)$ *be a Lie algebroid and* $\phi \in \Gamma(A^*)$ *a one-cocycle. There exists a unique operation* $[,]^\phi : \Gamma(\wedge^r A) \times \Gamma(\wedge^{r'} A) \to \Gamma(\wedge^{r+r'-1} A)$ *such that*

$$[X, f]^\phi = (\rho^\phi(X))f$$
$$[X, Y]^\phi = [X, Y]$$
$$[P, P']^\phi = -(-1)^{(r-1)(r'-1)}[P', P]^\phi$$
$$[P, P' \wedge P'']^\phi = [P, P']^\phi \wedge P'' + (-1)^{(r-1)r'} P' \wedge [P, P'']^\phi - (\iota_\phi P) \wedge P' \wedge P''$$

for $f \in C^\infty(M), X, Y \in \Gamma(A), P \in \Gamma(\wedge^r A), P' \in \Gamma(\wedge^{r'} A)$ *and* $P'' \in \Gamma(\wedge^{r''} A)$. *Furthermore, it satisfies the graded Jacobi identity*

$$\sum_{\text{cycl.}} (-1)^{(r-1)(r''-1)}[[P, P']^\phi, P'']^\phi = 0.$$

This operation is called the ϕ-*Schouten bracket* [17] or *Schouten–Jacobi bracket* [10] of $(A, [,], \rho)$. It is connected with the usual Schouten bracket by the formula

$$[P, P']^\phi = [P, P'] + (r-1)P \wedge (\iota_\phi P') - (-1)^{r-1}(r'-1)(\iota_\phi P) \wedge P'. \qquad (6)$$

The ϕ-Lie derivative $\mathcal{L}_X^\phi : \Gamma(\wedge^* A) \to \Gamma(\wedge^* A)$ with respect to $X \in \Gamma(A)$ is defined by

$$\mathcal{L}_X^\phi P = [X, P]^\phi$$

and it follows

$$\mathcal{L}_X^\phi [P, P']^\phi = \left[\mathcal{L}_X^\phi P, P'\right]^\phi + \left[P, \mathcal{L}_X^\phi P'\right]^\phi$$

for $P \in \Gamma(\wedge^r A)$ and $P' \in \Gamma(\wedge^{r'} A)$, which is an analogue of the usual Lie derivation. However, it is deduced that the following are deformed by ϕ.

Proposition 2.8.

$$\mathcal{L}_{fX}^\phi P = f\mathcal{L}_X^\phi P - X \wedge P(\mathrm{d}f)$$
$$\mathcal{L}_X^\phi (P \wedge P') = \mathcal{L}_X^\phi P \wedge P' + P \wedge \mathcal{L}_X^\phi P' - \phi(X) P \wedge P'$$
$$\mathcal{L}_X^\phi (P(\alpha)) = \left(\mathcal{L}_X^\phi P\right)(\alpha) + P\left(\mathcal{L}_X^\phi \alpha\right) + (k-1)\phi(X)P(\alpha) \qquad (k \leqslant r)$$
$$\mathcal{L}_X^\phi (\alpha(P)) = \left(\mathcal{L}_X^\phi \alpha\right)(P) + \alpha\left(\mathcal{L}_X^\phi P\right) + (r-1)\phi(X)\alpha(P) \qquad (r \leqslant k)$$

for $f \in C^\infty(M), X \in \Gamma(A), P \in \Gamma(\wedge^r A), P' \in \Gamma(\wedge^{r'} A)$ *and* $\alpha \in \Gamma(\wedge^k A^*)$.

Now, we will consider the Lie algebroid $(A \times \mathbb{R}, [,], \tilde{\rho})$ given by proposition 2.5. It follows that $(0, 1) \in \Gamma(A^*) \times C^\infty(M) \simeq \Gamma(A^* \times \mathbb{R})$ is a one-cocycle. Applying (6) to this case, we have the following propositions for Lie algebroid Jacobi structures.

Proposition 2.9 [17]**.** *Let* $(A, [,], \rho)$ *be a Lie algebroid. A two-section* $(\pi, E) \in \Gamma(\wedge^2(A \times \mathbb{R}))$ *of* $A \times \mathbb{R}$ *is a Jacobi structure for* $(A, [,], \rho)$ *if and only if*

$$[(\pi, E), (\pi, E)]^{(0,1)} = 0.$$

Proposition 2.10 [17]**.** *Suppose that* $(\pi, E) \in \Gamma(\wedge^2(A \times \mathbb{R}))$ *is a Jacobi structure for a Lie algebroid* $(A, [,], \rho)$. *Then it associates the Lie algebroid structure on* $A^* \times \mathbb{R}$ *whose bracket and anchor are written respectively as*

$$[\boldsymbol{\alpha}, \boldsymbol{\beta}] = \mathcal{L}_{\boldsymbol{\pi}(\boldsymbol{\alpha})}^{(0,1)} \boldsymbol{\beta} - \mathcal{L}_{\boldsymbol{\pi}(\boldsymbol{\beta})}^{(0,1)} \boldsymbol{\alpha} - d^{(0,1)}(\boldsymbol{\pi}(\boldsymbol{\alpha}, \boldsymbol{\beta}))$$
$$\rho_{(\pi,E)} = \tilde{\rho} \circ \boldsymbol{\pi}$$

where $\boldsymbol{\pi} = (\pi, E)$, $\boldsymbol{\alpha} = (\alpha, f)$, $\boldsymbol{\beta} = (\beta, g)$ *and* $\tilde{\rho}$ *is the anchor of* $A \times \mathbb{R}$ *(see proposition 2.5). Moreover,*

$$\begin{array}{ccc} \Gamma(A^* \times \mathbb{R}) & \xrightarrow{\ \pi\ } & \Gamma(A \times \mathbb{R}) \\ {\scriptstyle \rho_{(\pi,E)}}\downarrow & \swarrow \tilde{\rho} & \\ \Gamma(TM) & & \end{array}$$

is a commutative diagram of Lie algebra homomorphisms.

By proposition 2.10, we recover the Lie algebroid structure on $T^*M \times \mathbb{R}$ associated with a Jacobi manifold (M, π, E) given in [19] as

$$[(\alpha, f), (\beta, g)] = (\mathcal{L}_{\pi(\alpha)}\beta - \mathcal{L}_{\pi(\beta)}\alpha - \mathrm{d}(\pi(\alpha, \beta)) + f\mathcal{L}_E\beta - g\mathcal{L}_E\alpha - \iota_E(\alpha \wedge \beta),$$
$$\pi(\beta, \alpha) + (\pi(\alpha))g - (\pi(\beta))f + fEg - gEf)$$
$$\rho_{(\pi,E)}(\alpha, f) = \rho(\pi(\alpha) + fE)$$

for $(\alpha, f), (\beta, g) \in \Gamma(A^*) \times C^\infty(M)$.

It may be seen that a Jacobi structure for a Lie algebroid A is formally interpreted as a 'Poisson structure' on $A \times \mathbb{R}$ in the presence of the one-cocycle $(0, 1)$. In fact, the pair of a Lie algebroid and a one-cocycle is equivalent to the notion of *Jacobi algebroid* [10], and d^ϕ, $\mathcal{L}^\phi$, $[,]^\phi$ above are defined as natural ones on it. Therefore, a Jacobi structure on a Lie algebroid is formally considered a 'Poisson structure' for the Jacobi algebroid.

3. Nambu–Jacobi structures in terms of Jacobi algebroids

3.1. Nambu–Jacobi structures revisited

Now, we will describe Nambu–Jacobi structures and associated Leibniz algebroids using the notion of Jacobi algebroid, as in the case of Jacobi structures.

We will treat a Nambu–Jacobi structure of order p as a p-section of $TM \times \mathbb{R}$. The right 'deformation' is obtained by taking the one-cocycle $(0, p-1) \in \Gamma(T^*M \times \mathbb{R})$. First, we show the condition for $\mathbf{\Pi} \in \Gamma(\wedge^p(TM \times \mathbb{R})) \simeq \Gamma(\wedge^p TM) \oplus \Gamma(\wedge^{p-1}TM)$ to be a Nambu–Jacobi structure.

Proposition 3.1. *Suppose that* $p \geqslant 3$. *Then* $\mathbf{\Pi} \in \Gamma(\wedge^p(TM \times \mathbb{R}))$ *is a Nambu–Jacobi structure of order p on M if and only if*

$$[\mathbf{\Pi}(\boldsymbol{\alpha}), \mathbf{\Pi}]^\phi = (-1)^p(\mathbf{\Pi}(d^\phi\boldsymbol{\alpha} + (p-2)\phi \wedge \boldsymbol{\alpha}))\mathbf{\Pi} \tag{7}$$

for any $\boldsymbol{\alpha} \in \Gamma(\wedge^{p-1}(T^*M \times \mathbb{R}))$, *where* $\phi = (0, p-1) \in \Gamma(T^*M \times \mathbb{R})$.

Proof. Put $\mathbf{\Pi} = (\Pi, E)$ and $\boldsymbol{\alpha} = (\alpha, \beta)$. Suppose that $\mathbf{\Pi}$ is a Nambu–Jacobi structure. Since (1)–(4) and

$$d^\phi(\alpha, \beta) = (\mathrm{d}\alpha, (p-1)\alpha - \mathrm{d}\beta)$$

we calculate

$$\begin{aligned}[\mathbf{\Pi}(\boldsymbol{\alpha}), \mathbf{\Pi}]^\phi &= [\mathbf{\Pi}(\boldsymbol{\alpha}), \mathbf{\Pi}] - (p-1)(\phi(\mathbf{\Pi}(\boldsymbol{\alpha})))\mathbf{\Pi} \\ &= ([\Pi(\alpha) + E(\beta), \Pi], [\Pi(\alpha) + E(\beta), E] - [(-1)^{p-1}E(\alpha), \Pi]) \\ &\quad + (-1)^p(p-1)\mathbf{\Pi}(\phi \wedge \boldsymbol{\alpha})\mathbf{\Pi} \\ &= (-1)^p((\Pi, E)(\mathrm{d}(\alpha, \beta)))(\Pi, E) + (-1)^p(p-1)\mathbf{\Pi}(\phi \wedge \boldsymbol{\alpha})\mathbf{\Pi} \\ &= (-1)^p(\mathbf{\Pi}(d^\phi\boldsymbol{\alpha} + (p-2)\phi \wedge \boldsymbol{\alpha}))\mathbf{\Pi}.\end{aligned}$$

Conversely, it follows easily that we have the definition (1)–(4) from (7). □

Equation (7) also gives an equivalent definition for a locally decomposable Jacobi structure $\mathbf{\Pi}$, that is, $\mathbf{\Pi} = \boldsymbol{X}_1 \wedge \boldsymbol{X}_2$ locally for some $\boldsymbol{X}_1, \boldsymbol{X}_2 \in \Gamma(TM \times \mathbb{R})$ around a point where $\mathbf{\Pi} \neq 0$. We will see later that an arbitrary Nambu–Jacobi structure is locally decomposable.

Recall that the bundle $TM \times \mathbb{R}$ is a Lie algebroid and $\tilde{\rho}$ denotes its anchor (see proposition 2.5). Theorem 2.4 is written as follows:

Theorem 3.2. *Let $\mathbf{\Pi} \in \Gamma(\wedge^p(TM \times \mathbb{R}))$ be a Nambu–Jacobi structure of order $p \geqslant 2$, from which we assume $\mathbf{\Pi}$ is locally decomposable as a section of $\wedge^2(TM \times \mathbb{R})$ when $p = 2$. Then the triple $(\wedge^{p-1}T^*M \oplus \wedge^{p-2}T^*M, [\![,]\!], \rho_{\mathbf{\Pi}})$ is a Leibniz algebroid where $\rho_{\mathbf{\Pi}} = \tilde{\rho} \circ \mathbf{\Pi}$ and*

$$[\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!] = \mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}\boldsymbol{\beta} + (-1)^p(\mathbf{\Pi}(d^{\phi}\boldsymbol{\alpha}))\boldsymbol{\beta}$$

for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Gamma(\wedge^{p-1}(T^*M \times \mathbb{R}))$, where $\phi = (0, p-1) \in \Gamma(T^*M \times \mathbb{R})$. Conversely, any p-section $\mathbf{\Pi}$ which gives $\wedge^{p-1}(T^*M \times \mathbb{R})$ a Leibniz algebroid structure above is necessarily a Nambu–Jacobi structure on M.

Proof. We will verify that it corresponds with theorem 2.4 first, and then give an alternative proof using (7).

It is easily seen that the anchor $\rho_{\mathbf{\Pi}}$ corresponds with $\rho_{(\Pi,E)}$. We will see that brackets agree. Put $\mathbf{\Pi} = (\Pi, E)$, $\boldsymbol{\alpha} = (\alpha, \alpha')$ and $\boldsymbol{\beta} = (\beta, \beta')$ respectively. Using the calculus prepared in section 2.2, we compute

$$\begin{aligned}\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}\boldsymbol{\beta} = (&\mathcal{L}_{\Pi(\alpha)}\beta + \mathcal{L}_{E(\alpha')}\beta + (-1)^{p-1}\,\mathrm{d}(E(\alpha)) \wedge \beta' + (-1)^{p-1}(p-1)(E(\alpha))\beta,\\ &\mathcal{L}_{\Pi(\alpha)}\beta' + (-1)^{p-1}(E(\alpha))\beta')\end{aligned}$$

and

$$(\mathbf{\Pi}(d^{\phi}\boldsymbol{\alpha}))\boldsymbol{\beta} = (\Pi(\mathrm{d}\alpha) + (p-1)E(\alpha), -E(\mathrm{d}\alpha'))(\beta, \beta')$$

from which we recover the bracket in theorem 2.4.

Now we will give an alternative proof. From (7) it follows

$$\begin{aligned}\mathbf{\Pi}([\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]) &= [\mathbf{\Pi}(\boldsymbol{\alpha}), \mathbf{\Pi}(\boldsymbol{\beta})] - \big(\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}\mathbf{\Pi}\big)\boldsymbol{\beta} - (p-2)\mathbf{\Pi}(\boldsymbol{\alpha} \wedge \phi)\mathbf{\Pi}(\boldsymbol{\beta})\\ &\quad + (-1)^p(\mathbf{\Pi}(d^{\phi}\boldsymbol{\alpha}))\mathbf{\Pi}(\boldsymbol{\beta})\\ &= [\mathbf{\Pi}(\boldsymbol{\alpha}), \mathbf{\Pi}(\boldsymbol{\beta})].\end{aligned}$$

We also have

$$[\![\boldsymbol{\alpha}, f\boldsymbol{\beta}]\!] = (\tilde{\rho}(\mathbf{\Pi}(\boldsymbol{\alpha}))f)\boldsymbol{\beta} + f[\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]$$

thus we must check the Leibniz identity. Using the calculus prepared in section 2.2, for $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \Gamma(\wedge^{p-1}(TM \times \mathbb{R}))$ we calculate

$$\begin{aligned}[\![\boldsymbol{\alpha}, [\![\boldsymbol{\beta}, \boldsymbol{\gamma}]\!]]\!] &= \mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\beta})}\boldsymbol{\gamma} + (-1)^p(\mathcal{L}_{\mathbf{\Pi}(\boldsymbol{\alpha})}(\mathbf{\Pi}(d^{\phi}\boldsymbol{\beta})))\boldsymbol{\gamma} + (-1)^p(\mathbf{\Pi}(d^{\phi}\boldsymbol{\beta}))\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}\boldsymbol{\gamma}\\ &\quad + (-1)^p(\mathbf{\Pi}(d^{\phi}\boldsymbol{\alpha}))\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\beta})}\boldsymbol{\gamma} + (\mathbf{\Pi}(d^{\phi}\boldsymbol{\alpha}))(\mathbf{\Pi}(d^{\phi}\boldsymbol{\beta}))\boldsymbol{\gamma}\\ [\![[\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!], \boldsymbol{\gamma}]\!] &= \mathcal{L}^{\phi}_{[\mathbf{\Pi}(\boldsymbol{\alpha}),\mathbf{\Pi}(\boldsymbol{\beta})]}\boldsymbol{\gamma} + (-1)^p\big(\mathbf{\Pi}\big(\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}d^{\phi}\boldsymbol{\beta}\big)\big)\boldsymbol{\gamma}\\ &\quad + \mathbf{\Pi}(d(\mathbf{\Pi}(d^{\phi}\boldsymbol{\alpha})) \wedge \boldsymbol{\beta})\boldsymbol{\gamma} + (\mathbf{\Pi}(d^{\phi}\boldsymbol{\alpha}))(\mathbf{\Pi}(d^{\phi}\boldsymbol{\beta}))\boldsymbol{\gamma}\\ &= \mathcal{L}^{\phi}_{[\mathbf{\Pi}(\boldsymbol{\alpha}),\mathbf{\Pi}(\boldsymbol{\beta})]}\boldsymbol{\gamma} + (-1)^p(\mathcal{L}_{\mathbf{\Pi}(\boldsymbol{\alpha})}(\mathbf{\Pi}(d^{\phi}\boldsymbol{\beta})))\boldsymbol{\gamma} - (-1)^p(\mathcal{L}_{\mathbf{\Pi}(\boldsymbol{\beta})}(\mathbf{\Pi}(d^{\phi}\boldsymbol{\alpha})))\boldsymbol{\gamma}\\ [\![\boldsymbol{\beta}, [\![\boldsymbol{\alpha}, \boldsymbol{\gamma}]\!]]\!] &= \mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\beta})}\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}\boldsymbol{\gamma}\\ &\quad + (-1)^p(\mathcal{L}_{\mathbf{\Pi}(\boldsymbol{\beta})}(\mathbf{\Pi}(d^{\phi}\boldsymbol{\alpha})))\boldsymbol{\gamma} + (-1)^p(\mathbf{\Pi}(d^{\phi}\boldsymbol{\alpha}))\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\beta})}\boldsymbol{\gamma}\\ &\quad + (-1)^p(\mathbf{\Pi}(d^{\phi}\boldsymbol{\beta}))\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}\boldsymbol{\gamma} + (\mathbf{\Pi}(d^{\phi}\boldsymbol{\beta}))(\mathbf{\Pi}(d^{\phi}\boldsymbol{\alpha}))\boldsymbol{\gamma}\end{aligned}$$

and we have the Leibniz identity. □

Example 3.3. Let $f_1, \ldots, f_{p-1}, g_1, \ldots, g_{p-1}$ be functions on a Nambu–Jacobi manifold $(M, \{, \ldots, \})$ of order p. We have

$$[\![d^{(0,1)} f_1 \wedge \cdots \wedge d^{(0,1)} f_{p-1}, d^{(0,1)} g_1 \wedge \cdots \wedge d^{(0,1)} g_{p-1}]\!] = \sum_{i=1}^{p-1} d^{(0,1)} g_1 \wedge \cdots \wedge d^{(0,1)} \{f_1, \ldots, f_{p-1}, g_i\} \wedge \cdots \wedge d^{(0,1)} g_{p-1}$$

which formally corresponds to the Leibniz bracket for powers of the differentials of functions on a Nambu–Poisson manifold (see [28]).

3.2. Alternative Leibniz algebroids associated with Nambu–Jacobi structures

We have shown that the Leibniz algebroid associated with a Nambu–Jacobi manifold given by theorem 2.4 formally corresponds to that for a Nambu–Poisson manifold. For a Nambu–Poisson manifold, another associated Leibniz algebroid structure on the same bundle was given in [11], which is the natural generalization of the Lie algebroid structure on the cotangent bundle associated with a Poisson manifold. We will elucidate that a Nambu–Jacobi structure has the corresponding associated Leibniz algebroid structures.

First, we will see that a Nambu–Jacobi structure $\boldsymbol{\Pi}$ of order $p \geqslant 3$ on a manifold M is locally decomposable. When $\boldsymbol{\Pi} = (\Pi, E) \neq 0$ at a point on M, either $\Pi \neq 0$ or $E \neq 0$ holds locally around this point. At the point where $\Pi \neq 0$, we may write $\boldsymbol{\Pi} = (\Pi, \Pi(\theta))$ locally for some 1-form θ (see [27]), and since $\Pi = X_1 \wedge \cdots \wedge X_p$ for some vector fields $X_1, \ldots, X_p$ on M (recall that Π is a Nambu–Poisson structure), we have

$$(\Pi, \Pi(\theta)) = (X_1, X_1(\theta)) \wedge \cdots \wedge (X_p, X_p(\theta)).$$

On the other hand, at the point where $E \neq 0$ we have

$$\boldsymbol{\Pi} = (v \wedge E, E) = (v, 1) \wedge (E, 0)$$

for some vector field v (see also [27]) and $(E, 0)$ is locally decomposable at this point since E is. We have shown

Proposition 3.4. *A Nambu–Jacobi structure $\boldsymbol{\Pi} \in \Gamma(\wedge^p(TM \times \mathbb{R}))$ of order $p \geqslant 3$ on a manifold M is locally decomposable, that is, at a point where $\boldsymbol{\Pi} \neq 0$ there exist $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_p \in \Gamma(TM \times \mathbb{R})$ such that*

$$\boldsymbol{\Pi} = \boldsymbol{X}_1 \wedge \cdots \wedge \boldsymbol{X}_p$$

locally around this point.

The decomposability ensures the following.

Lemma 3.5. *Let $\boldsymbol{\Pi}$ be a Nambu–Jacobi structure of order p on M which we assume locally decomposable when $p = 2$. Then*

$$\boldsymbol{\Pi}(\iota_{\boldsymbol{\Pi}(\boldsymbol{\beta})} d^{\phi} \boldsymbol{\alpha}) = (-1)^{p-1} (\boldsymbol{\Pi}(d^{\phi} \boldsymbol{\alpha})) \boldsymbol{\Pi}(\boldsymbol{\beta}) \tag{8}$$

for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Gamma(\wedge^{p-1}(T^*M \times \mathbb{R}))$.

Proof. For any $\boldsymbol{\gamma} \in \Gamma(T^*M \times \mathbb{R})$, we have

$$(-1)^{p-1} (\boldsymbol{\Pi}(\iota_{\boldsymbol{\Pi}(\boldsymbol{\beta})} d^{\phi} \boldsymbol{\alpha}))(\boldsymbol{\gamma}) = \boldsymbol{\Pi}(\boldsymbol{\gamma} \wedge \iota_{\boldsymbol{\Pi}(\boldsymbol{\beta})} d^{\phi} \boldsymbol{\alpha}) = (\boldsymbol{\Pi}(\boldsymbol{\gamma}))(\iota_{\boldsymbol{\Pi}(\boldsymbol{\beta})} d^{\phi} \boldsymbol{\alpha})$$
$$= (\boldsymbol{\Pi}(\boldsymbol{\beta}) \wedge \boldsymbol{\Pi}(\boldsymbol{\gamma}))(d^{\phi} \boldsymbol{\alpha}) = (\boldsymbol{\Pi}(\boldsymbol{\beta} \wedge \boldsymbol{\gamma})) \boldsymbol{\Pi}(d^{\phi} \boldsymbol{\alpha}) = (\boldsymbol{\Pi}(d^{\phi} \boldsymbol{\alpha}))(\boldsymbol{\Pi}(\boldsymbol{\beta}))(\boldsymbol{\gamma}). \qquad \square$$

Now we will see that another Leibniz algebroid structure is associated with a Nambu–Jacobi structure of order p.

Theorem 3.6. *Let $\mathbf{\Pi} \in \Gamma(\wedge^p(TM \times \mathbb{R}))$ be a Nambu–Jacobi structure of order p on M. Then the triple $(\wedge^{p-1}T^*M \oplus \wedge^{p-2}T^*M, [\![,]\!]', \rho_{\mathbf{\Pi}})$ is a Leibniz algebroid where $\rho_{\mathbf{\Pi}} = \tilde{\rho} \circ \mathbf{\Pi}$ and*

$$[\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]' = \mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}\boldsymbol{\beta} - \iota_{\mathbf{\Pi}(\boldsymbol{\beta})} d^{\phi}\boldsymbol{\alpha}$$

for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Gamma(\wedge^{p-1}(T^*M \times \mathbb{R}))$, where $\phi = (0, p-1) \in \Gamma(T^*M \times \mathbb{R})$ and $\tilde{\rho}$ is the anchor of $TM \times \mathbb{R}$. Conversely, any p-section $\mathbf{\Pi}$ which gives $\wedge^{p-1}(T^*M \times \mathbb{R})$ a Leibniz algebroid structure above is necessarily a Nambu–Jacobi structure on M.

Proof. We have proposition 2.10 for $p = 2$. Suppose $p \geqslant 3$. The derivation law

$$[\![\boldsymbol{\alpha}, f\boldsymbol{\beta}]\!]' = (\tilde{\rho}(\mathbf{\Pi}(\boldsymbol{\alpha}))f)\boldsymbol{\beta} + f[\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]'$$

is proved as in theorem 3.2. It follows from lemma 3.5 that the last sentence of the theorem and that $\rho_{\mathbf{\Pi}}$ induces a Leibniz algebra homomorphism

$$\rho_{\mathbf{\Pi}}([\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]') = [\rho_{\mathbf{\Pi}}(\boldsymbol{\alpha}), \rho_{\mathbf{\Pi}}(\boldsymbol{\beta})]$$

also hold as in theorem 3.2. We calculate the Leibniz identity. For all $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \Gamma(\wedge^{p-1}(TM \times \mathbb{R})$, we have

$$\begin{aligned}
[\![\boldsymbol{\alpha}, [\![\boldsymbol{\beta}, \boldsymbol{\gamma}]\!]']\!]' &= \mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}\big(\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\beta})}\boldsymbol{\gamma} - \iota_{\mathbf{\Pi}(\boldsymbol{\gamma})}d^{\phi}\boldsymbol{\beta}\big) - \iota_{\mathbf{\Pi}([\![\boldsymbol{\beta},\boldsymbol{\gamma}]\!]')}d^{\phi}\boldsymbol{\alpha} \\
&= \mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\beta})}\boldsymbol{\gamma} - \mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}(d^{\phi}\boldsymbol{\beta}(\mathbf{\Pi}(\boldsymbol{\gamma}))) - d^{\phi}\boldsymbol{\alpha}\big(\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\beta})}(\mathbf{\Pi}(\boldsymbol{\gamma}))\big) \\
&= \mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\beta})}\boldsymbol{\gamma} - \big(\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}d^{\phi}\boldsymbol{\beta}\big)(\mathbf{\Pi}(\boldsymbol{\gamma})) \\
&\quad - d^{\phi}\boldsymbol{\beta}\big(\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}(\mathbf{\Pi}(\boldsymbol{\gamma}))\big) - d^{\phi}\boldsymbol{\alpha}\big(\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\beta})}(\mathbf{\Pi}(\boldsymbol{\gamma}))\big) \\
[\![[\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]', \boldsymbol{\gamma}]\!]' &= \mathcal{L}^{\phi}_{[\mathbf{\Pi}(\boldsymbol{\alpha}),\mathbf{\Pi}(\boldsymbol{\beta})]}\boldsymbol{\gamma} - \iota_{\mathbf{\Pi}(\boldsymbol{\gamma})}d^{\phi}\big(\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}\boldsymbol{\beta} - \iota_{\mathbf{\Pi}(\boldsymbol{\beta})}d^{\phi}\boldsymbol{\alpha}\big) \\
&= \mathcal{L}^{\phi}_{[\mathbf{\Pi}(\boldsymbol{\alpha}),\mathbf{\Pi}(\boldsymbol{\beta})]}\boldsymbol{\gamma} - \big(\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}d^{\phi}\boldsymbol{\beta}\big)(\mathbf{\Pi}(\boldsymbol{\gamma})) + \big(\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\beta})}d^{\phi}\boldsymbol{\alpha}\big)(\mathbf{\Pi}(\boldsymbol{\gamma})) \\
[\![\boldsymbol{\beta}, [\![\boldsymbol{\alpha}, \boldsymbol{\gamma}]\!]']\!]' &= \mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\beta})}\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}\boldsymbol{\gamma} - \big(\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\beta})}d^{\phi}\boldsymbol{\alpha}\big)(\mathbf{\Pi}(\boldsymbol{\gamma})) \\
&\quad - d^{\phi}\boldsymbol{\alpha}\big(\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\beta})}(\mathbf{\Pi}(\boldsymbol{\gamma}))\big) - d^{\phi}\boldsymbol{\beta}\big(\mathcal{L}^{\phi}_{\mathbf{\Pi}(\boldsymbol{\alpha})}(\mathbf{\Pi}(\boldsymbol{\gamma}))\big).
\end{aligned}$$

Thus we get the Leibniz identity. □

It is a natural generalization of the Lie algebroid structure on the bundle of 1-jets associated with a Jacobi manifold (see proposition 2.10). Moreover, when $\mathbf{\Pi}$ defines a Nambu–Poisson structure, that is, $\mathbf{\Pi} = (\Pi, 0)$, we obtain the Leibniz algebroid structure on the bundle of $(p-1)$-forms given in [11] as

$$[\![\alpha, \beta]\!]' = \mathcal{L}_{\Pi(\alpha)}\beta - \iota_{\Pi(\beta)}\,\mathrm{d}\alpha \qquad \rho_{\Pi}(\alpha) = \Pi(\alpha)$$

for $\alpha, \beta \in \Gamma(\wedge^{p-1}T^*M)$. We remark that, as in the case of $p = 2$, it follows that

$$\begin{array}{ccc}
\Gamma(\wedge^{p-1}(T^*M \times \mathbb{R})) & \xrightarrow{\ \mathbf{\Pi}\ } & \Gamma(TM \times \mathbb{R}) \\
\rho_{\mathbf{\Pi}} \downarrow & & \swarrow \tilde{\rho} \\
\Gamma(TM) & &
\end{array}$$

is a commutative diagram of Leibniz algebra homomorphisms. We also remark that it is not a matched pair of Leibniz algebroids as in theorem 2.4.

Example 3.7. Let $\mathbf{\Pi} \in \Gamma(\wedge^p(TM \times \mathbb{R}))$ be a Nambu–Jacobi structure of order p on M. For $d^{(0,p-1)}$-closed sections $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Gamma(\wedge^{p-1}(T^*M \times \mathbb{R}))$, it follows $[\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]' = [\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]$. Since

$$d^{(0,1)}f_1 \wedge \cdots \wedge d^{(0,1)}f_{p-1} = d^{(0,p-1)}\boldsymbol{f}$$

where

$$\boldsymbol{f} = \left(\frac{1}{p-1} \sum_{i=1}^{p} (-1)^{i-1} f_i df_1 \wedge \cdots \widehat{df_i} \cdots \wedge df_{p-1}, 0 \right)$$

we have

$$\begin{aligned}&[\![d^{(0,1)} f_1 \wedge \cdots \wedge d^{(0,1)} f_{p-1}, d^{(0,1)} g_1 \wedge \cdots \wedge d^{(0,1)} g_{p-1}]\!]' \\ &\qquad = \sum_{i=1}^{p-1} d^{(0,1)} g_1 \wedge \cdots \wedge d^{(0,1)} \{f_1, \ldots, f_{p-1}, g_i\} \wedge \cdots \wedge d^{(0,1)} g_{p-1}\end{aligned}$$

for functions $f_1, \ldots, f_{p-1}, g_1, \ldots, g_{p-1}$.

In general, the two Leibniz algebroid structures do not agree.

We also have the Nambu–Jacobi counterpart of another theorem given in [11]:

Theorem 3.8. *Let $\boldsymbol{\Pi} \in \Gamma(\wedge^p(TM \times \mathbb{R}))$ be a Nambu–Jacobi structure of order p on M. Then the triple $(\wedge^{p-1}(T^*M \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}), [\![,]\!]'', \rho''_{\boldsymbol{\Pi}})$ is a Leibniz algebroid over M where $[\![,]\!]''$ and $\rho''_{\boldsymbol{\Pi}}$ are defined respectively by*

$$[\![(\boldsymbol{\alpha}, \boldsymbol{\alpha}'), (\boldsymbol{\beta}, \boldsymbol{\beta}')]\!]'' = \left(\mathcal{L}^{\phi}_{\boldsymbol{\Pi}(\boldsymbol{\alpha})} \boldsymbol{\beta} - \iota_{\boldsymbol{\Pi}(\boldsymbol{\beta})} d^{\phi} \boldsymbol{\alpha}, \mathcal{L}^{\phi}_{\boldsymbol{\Pi}(\boldsymbol{\alpha})} \boldsymbol{\beta}' - \iota_{\boldsymbol{\Pi}(\boldsymbol{\beta}')} d^{\phi} \boldsymbol{\alpha} \right)$$

$$\rho''_{\boldsymbol{\Pi}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}') = \tilde{\rho} \circ \boldsymbol{\Pi}(\boldsymbol{\alpha})$$

for $(\boldsymbol{\alpha}, \boldsymbol{\alpha}'), (\boldsymbol{\beta}, \boldsymbol{\beta}') \in \Gamma(\wedge^{p-1}(T^*M \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}))$, where $\phi = (0, p-1) \in \Gamma(T^*M \times \mathbb{R})$ and $\tilde{\rho}$ is the anchor of $TM \times \mathbb{R}$. Moreover,

$$\begin{array}{ccc} \Gamma(\wedge^{p-1}(T^*M \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})) & \xrightarrow{\mathrm{pr}_1} & \Gamma(\wedge^{p-1}(T^*M \times \mathbb{R})) \\ \rho''_{\boldsymbol{\Pi}} \big\downarrow & \swarrow \rho_{\boldsymbol{\Pi}} & \\ \Gamma(TM) & & \end{array}$$

is a commutative diagram of Leibniz algebra homomorphisms where $\mathrm{pr}_1 : \wedge^{p-1}(T^*M \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}) \to \wedge^{p-1}(T^*M \times \mathbb{R})$ is the natural projection and the Leibniz algebroid structure on $\wedge^{p-1}(T^*M \times \mathbb{R})$ is given by theorem 3.6.

Proof. We only need the Leibniz identity for $\Gamma(T^*M \times \mathbb{R})$ components, which is proved by formally replacing $\boldsymbol{\alpha}, \boldsymbol{\beta}$ with $\boldsymbol{\alpha}', \boldsymbol{\beta}'$ respectively in the proof of the Leibniz identity in theorem 3.6. □

Finally, we discuss the generalization of our results from on a manifold to on a Lie algebroid. When we define a Nambu–Poisson structure Π of order $p \geqslant 3$ on a Lie algebroid A as a p-section Π of A satisfying $[\Pi(\alpha), \Pi] = (-1)^p (\Pi(d\alpha))\Pi$ for any $\alpha \in \Gamma(\wedge^{p-1} A^*)$, the Leibniz algebroid structure given in [15] (the case of a Nambu–Poisson structure in theorem 2.4) is generalized to the case of a Lie algebroid. However, to generalize those given in [11] (the cases of a Nambu–Poisson structure in theorems 3.6 and 3.8), we need the decomposability of Π which is obscured and different from the case on a manifold. A sufficient condition for the decomposability is given as follows:

Lemma 3.9. *Suppose that $\Gamma(A^*)$ is generated by the differentials of functions. Then a Nambu–Poisson structure Π of order $p \geqslant 3$ on A is locally decomposable to sections of A*

Proof. Computing $[\Pi(f\alpha), \Pi]$ where f is a function, we have $(\Pi(\alpha \wedge \gamma))\Pi = \Pi(\alpha) \wedge \Pi(\gamma)$ for any $\alpha \in \Gamma(\wedge^{p-1} A^*)$ and $\gamma \in \Gamma(A^*)$. □

In [33], there is a different definition of a Nambu–Poisson structure on a Lie algebroid A; it is defined as a p-section Π of A satisfying

$$[\Pi(\alpha), \Pi](\beta) = -\Pi(\iota_{\Pi(\beta)}\, \mathrm{d}\alpha) \tag{9}$$

for any $\alpha, \beta \in \Gamma(\wedge^{p-1} A^*)$. It follows that the Leibniz algebroid structure given in [11] is generalized to the case of a Lie algebroid under this definition. However, to generalize that given in [15], we need the decomposability of Π which is not clear also in this definition; a sufficient condition for the decomposability has also been given by the lemma above. Both the definitions coincide if the decomposability is satisfied.

We have two definitions of the Nambu–Jacobi structure on a Lie algebroid as those of the Nambu–Poisson structure above. From one definition we deduce theorem 3.2, and from the other theorems 3.6 and 3.8. Both coincide if the decomposability is satisfied.

Although the necessity of decomposability makes it difficult to generalize our results to the case of a Lie algebroid, it seems to be worth considering a 'Nambu–Poisson structure' on an arbitrary Jacobi algebroid. We will define a (Nambu–)Poisson structure (of order 2) on a Jacobi algebroid (A, ϕ) as a 2-section $\mathbf{\Pi}$ of A satisfying $[\mathbf{\Pi}, \mathbf{\Pi}]^{\phi} = 0$ and a Nambu–Poisson structure of order $p \geqslant 3$ as a p-section $\mathbf{\Pi}$ of A which satisfies equation (7) for any $(p-1)$-section $\boldsymbol{\alpha}$. Then when $(A, \phi) = (T^*M, 0)$ we have the Nambu–Poisson structure on a manifold M, and when $(A, \phi) = (T^*M \times \mathbb{R}, (0, p-1))$, we have the Nambu–Jacobi structure on M. That is, it will give a general framework for the unified theory. Theorem 3.2 is generalized to the case of an arbitrary Jacobi algebroid. If $\mathbf{\Pi}$ is locally decomposable, equation (8) of lemma 3.5 holds for any $(p-1)$-sections $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, and thus theorems 3.6 and 3.8 are also generalized. We finally remark that we may have another definition of the Nambu–Poisson structure on a Jacobi algebroid using (9). The circumstances are similar.

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